Permanence for Delay Difference Nonautonomous Population Models (Dynamics of Functional Equations and Related Topics)

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Permanence for Delay Difference Nonautonomous Population Models

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1. Introduction

Consider the Ricker population model

$$x(n+1)=x(n)\exp[r-ax(n)], \quad n=0,1,2,\cdots \tag{1}$$

where $r>0$ and $a>0$, and the initial condition of (1) is given as $x(0)>0$. A known result for the permanence of (1) is the following: (cf. Hofbauer *et al* [2]):

Theorem A *System (1) is permanent.*

Let $x(n)$ be any solution of system (1). We say (1) is permanent if there exist $\xi>0$ and $\eta>0$, independent of $x(n)$, such that $\xi\leq x(n)\leq\eta$ for all large n. Permanence is one of the most important questions from a biological point of view, which guarantees the long term survival of species. It is well known that the dynamics of difference equations may be extremely complex, even for one species, since, for example, chaotic behavior can occur even in one dimensional equations. Theorem A says that (1) is permanent even if chaotic behavior occurs for large r.

We next consider the Lotka-Volterra discrete competition population model:

$$\begin{cases} x(n+1)=x(n)\exp[r_1-a_{11}x(n)-a_{12}y(n)] \\ y(n+1)=y(n)\exp[r_2-a_{21}x(n)-a_{22}y(n)], \quad n=0,1,2,\cdots \end{cases} \tag{2}$$

where $r_i>0$, $a_{ii}>0$ $(i=1,2)$, $a_{ij}\geq 0$ $(i\neq j)$. The initial condition of (2) is given as $x(0)>0$ and $y(0)>0$. A known result for the permanence of (2) is the following (cf. [2]):

Theorem B *System (2) is permanent if and only if*

$$r_2a_{11}-r_1a_{21}>0 \qquad \textit{and} \qquad r_1a_{22}-r_2a_{12}>0$$

hold.

We will now take some delays into consideration on systems (1) and (2). Consider the Ricker population model with a delay

$$x(n+1)=x(n)\exp[r-ax(n-k)], \tag{3}$$

where $r > 0$, $a > 0$, $k \in Z^+$ (Z^+: the nonnegative integer set) and the Lotka-Volterra discrete competition population model with delays

$$\begin{cases} x(n+1) = x(n)\exp[r_1 - a_{11}x(n-k_1) - a_{12}y(n-k_2)] \\ y(n+1) = y(n)\exp[r_2 - a_{21}x(n-l_1) - a_{22}y(n-l_2)], \end{cases} \tag{4}$$

where $k_i, l_i \in Z^+$, $r_i > 0$, $a_{ii} > 0$ $(i = 1, 2)$, and $a_{ij} \geq 0$ $(i \neq j)$. The following theorems hold, which generalize Theorems A and B, respectively:

Theorem 1. *System (3) is permanent for all $k \in Z^+$.*

Theorem 2. *(Saito et al [4]) System (4) is permanent for all $k_i, l_i \in Z^+$ $(i = 1, 2)$ if and only if*

$$r_2a_{11} - r_1a_{21} > 0 \qquad \textit{and} \qquad r_1a_{22} - r_2a_{12} > 0$$

hold.

As seen in the comparison between Theorems A and B and Theorems 1 and 2, we can see that the delays k, k_1, k_2, l_1, and l_2 have no effect on dynamics of the discrete population models (3) and (4) from the point of view of permanence, that is: delays are 'harmless' on the permanence of these systems. That fact leads to the following question: what other forms of (3) and (4) can have properties that delays don't have any effect on permanence? In this paper, to answer this question at least in part, we will try to generalize systems (3) and (4) and try to generalize Theorems 1 and 2 further.

2. Generalization of Theorem 1

In this section we will generalize Theorem 1. We consider the nonautonomous delay difference population model

$$x(n+1) = x(n)f(n, x_n), \qquad n = 0, 1, 2, \cdots, \tag{5}$$

where $x_n(s) = x(n+s)$ for $s = -\nu, -\nu+1, \cdots, -1, 0$ ($\nu \in Z^+$). The initial condition of (5) is given as

$$x(s) \geq 0, s = -\nu, -\nu+1, \cdots, 0\ ;\ x(0) > 0.$$

Define $X = \{\phi : \{-\nu, -\nu+1, \cdots, 0\} \to R_+\}$ and $f : Z^+ \times X \to R_+$. For each $\phi \in X$, the norm of ϕ is defined as $||\phi|| = \max\{|\phi(s)| \,\big|\, s = -\nu, -\nu+1, \cdots, -1, 0\}$, where $|\cdot|$ is any norm in R.

The assumption of the functional f is done as follows:

$(C1)$ There exist constants $\delta_1 > 1$, $1 > \delta_2 > 0$, $K_1 > 0$, and $K_2 > 0$ $(K_2 \geq K_1)$ such that for all sufficiently large n,

$$\begin{aligned} &x(n+s) \in [0, K_1],\ s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n) > \delta_1, \\ &x(n+s) \in [K_2, \infty), s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n) < \delta_2. \end{aligned} \tag{6}$$

$(C2)$ For each $\bar{x} > 0$, there exists an $1 > l(\bar{x}) > 0$ such that for all sufficiently large n,

$$||x_n|| \leq \bar{x} \Longrightarrow f(n, x_n) \geq l(\bar{x}). \tag{7}$$

$(C3)$ There exists an $M > 1$ such that for all sufficiently large n,

$$x_n \in X \Longrightarrow f(n, x_n) \leq M. \tag{8}$$

$(C1)$ assumes that the growth rate for small population is positive, while there is a self-crowding effect creating a negative growth rate at high population levels. $(C2)$ states that the negative fluctuation effect on the growth rate is limited for a limited population density. $(C3)$ assumes that there is an upper bound for the growth rate. These assumptions $(C1)$, $(C2)$, and $(C3)$ are natural ones for population models because equation (3) and equations which will be mentioned later satisfy $(C1)$, $(C2)$, and $(C3)$.

The following is the main result:

Theorem 3. *Let f satisfy $(C1)$ — $(C3)$. Then system (5) is permanent.*

(*Proof.*) It follows from $(C1)$ and $(C3)$ that there exists a sufficiently large number $N > 0$ such that the following $(C1)'$ and $(C3)'$:

$(C1)'$ There exist constants $\delta_1 > 1$, $1 > \delta_2 > 0$, $K_1 > 0$, and $K_2 > 0$ $(K_2 \geq K_1)$ such that for all $n \geq N$,

$$\begin{aligned} &x(n+s) \in [0, K_1],\ s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n) > \delta_1, \\ &x(n+s) \in [K_2, \infty), s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n) < \delta_2. \end{aligned} \tag{9}$$

$(C3)'$ There exists an $M > 1$ such that for all $n \geq N$,

$$x_n \in X \Longrightarrow f(n, x_n) \leq M. \tag{10}$$

We need to show that there are two positive constants (independent of $x(0)$) η_1 and η_2 such that $\eta_1 \leq x(n) \leq \eta_2$ for all large n (depending on $x(0)$).

We first show that we can choose $\eta_2 = K_2 M^{\nu+1}$. From the latter of (9), we see that there is an $N_0 > N$ such that

$$x(N_0) \leq K_2 \qquad \text{and} \qquad x(N_0 + 1) \leq \eta_2. \tag{11}$$

We only prove $x(N_0) \leq K_2$ since we obtain from (10),

$$x(N_0 + 1) \leq x(N_0)M \leq K_2 M \leq \eta_2$$

if $x(N_0) \leq K_2$. If not, we have $x(n) > K_2$ for all large n. Hence, from (9) and (5) we obtain for all large n,

$$x(n+1) \leq x(n)\delta_2.$$

This implies $x(n) \to 0$ as $n \to +\infty$, which contradicts $x(n) > K_2$ for all large n.

We will show that $x(n) \leq \eta_2$ for all large n. Otherwise, it follows from (11) that there exist N_1 and N_2 ; $N_2 > N_1 > N$, such that

$$\begin{aligned} &x(N_1) \leq K_2, \quad x(N_2) \leq \eta_2, \quad x(N_2+1) > \eta_2 \\ &x(n) \in [K_2, \eta_2] \quad \text{for } n = N_1+1, \cdots, N_2. \end{aligned} \tag{12}$$

Since (9) holds, we have for $n \geq N_1$,

$$x(n+1) \leq x(n)M \leq x(n-1)M^2 \leq \cdots \leq x(N_1)M^{n+1-N_1}. \tag{13}$$

When $n = N_2$ on (13), we obtain from (12)

$$\eta_2 < x(N_2+1) \leq x(N_1)M^{N_2+1-N_1} \leq K_2 M^{N_2+1-N_1},$$

which implies $\nu + 1 < N_2 + 1 - N_1$ since $\eta_2 = K_2 M^{\nu+1}$ ans $M > 1$. Thus we get

$$N_2 - N_1 > \nu.$$

Therefore, $x(N_2 + s) \in [K_2, \infty)$ for $s = -\nu, \cdots, 0$. By (9) we then have

$$f(N_2, x_{N_2}) < \delta_2 < 1$$

and hence

$$x(N_2+1) = x(N_2)f(N_2, x_{N_2}) < x(N_2),$$

which is a contradiction to (12).

Let $\eta_1 = K_1 l(\eta_2)^{\nu+1}$ where $l(\eta_2)$ is defined as in (7). We will now prove that $x(n) \geq \eta_1$ for all large n. Since $x(n) \leq \eta_2$ for all large n, we have from (7) that there exists an $\bar{N} > N$ such that for all $n \geq \bar{N}$,

$$f(n, x_n) \geq l(\eta_2). \tag{14}$$

Then, by the former of (9), we see that there is a large $\bar{N}_0 > \bar{N}$ such that

$$x(\bar{N}_0) \geq K_1 \quad \text{and} \quad x(\bar{N}_0+1) \geq \eta_1. \tag{15}$$

We only show that $x(\bar{N}_0) \geq K_1$ since we have from (14) that $x(\bar{N}_0+1) \geq x(\bar{N}_0)l(\eta_2) \geq \eta_1$ if $x(\bar{N}_0) \geq K_1$. If not, we have $x(n) < K_1$ for all large n. From (9) and (5) we obtain for all large n,

$$x(n+1) > x(n)\delta_1.$$

This implies $x(n) \to +\infty$ as $n \to +\infty$, which contradicts that $x(n) < K_1$ for all large n.

Assume that there exists a large $\bar{N}_3 > \bar{N}$ such that $x(\bar{N}_3) < \eta_1$. Then, it follows from (15) that there exist $\bar{N}_1$ and $\bar{N}_2$; $\bar{N}_2 > \bar{N}_1 > \bar{N}$ such that

$$\begin{aligned} &x(\bar{N}_1) \geq K_1, \quad x(\bar{N}_2) \geq \eta_1, \quad x(\bar{N}_2+1) < \eta_1 \\ &x(n) \in [\eta_1, K_1] \quad \text{for } n = \bar{N}_1+1, \cdots, \bar{N}_2. \end{aligned} \tag{16}$$

Since (14) holds, we have for $n \geq \bar{N}_1$,

$$x(n+1) \geq x(n)l(\eta_2) \geq x(n-1)l(\eta_2)^2 \geq \cdots \geq x(\bar{N}_1)l(\eta_2)^{n+1-\bar{N}_1}.$$

Letting $n = \bar{N_2}$, we obtain from (16)

$$\eta_1 > x(\bar{N_2}+1) \geq x(\bar{N_1})l(\eta_2)^{\bar{N_2}+1-\bar{N_1}} \geq K_1 l(\eta_2)^{\bar{N_2}+1-\bar{N_1}},$$

which implies $\nu + 1 < \bar{N_2} + 1 - \bar{N_1}$ since $\eta_1 = K_1 l(\eta_2)^{\nu+1}$. Thus we get

$$\bar{N_2} - \bar{N_1} > \nu.$$

Therefore, $x(\bar{N_2}+s) \in [0, K_1)$ for $s = -\nu, \cdots, 0$. By (9), we must have

$$x(\bar{N_2}+1) = x(\bar{N_2})f(\bar{N_2}, x_{\bar{N_2}}) > x(\bar{N_2})\delta_1 > x(\bar{N_2}),$$

which is contradiction to (16). This completes the proof.

3. Corollaries of Theorem 3

(I) Consider the Ricker difference equation with a finite number of delays

$$x(n+1) = x(n)\exp\left[r - \sum_{j=1}^{m} a_j x(n-k_j)\right] \tag{17}$$

where r and a_j are constants with $r > 0$, $a_j \geq 0$, and $\sum_{j=1}^{m} a_j > 0$, and $k_j \in Z^+$ $(j = 1, \cdots, m)$. Define

$$f(n, x_n) = \exp\bigl[r - \sum_{j=1}^{m} a_j x_n(-k_j)\bigr].$$

Then $(C1)$ — $(C3)$ are satisfied. By Theorem 3, the following holds:

Corollary 1. *System (17) is permanent for all $k_j \in Z^+$ $(j = 1, \cdots, m)$.*

It is clear that Corollary 1 generalizes Theorem 1.

(II) Consider the nonautonomous Ricker difference equation with a finite number of delays

$$x(n+1) = x(n)\exp\left[r(n) - \sum_{j=1}^{m} a_j(n) x(n-k_j)\right]. \tag{18}$$

Here $r(n)$ and $a_j(n)$ are functions with

$$0 < \rho_1 \leq \liminf r(n) \leq \limsup r(n) \leq \rho_2,$$
$$0 \leq \alpha_{1j} \leq \liminf a_j(n) \leq \limsup a_j(n) \leq \alpha_{2j},$$

where ρ_1, ρ_2, α_{1j}, and α_{2j} are constants with $\sum_{j=1}^{m} \alpha_{1j} > 0$, and $k_j \in Z^+$ $(j = 1, \cdots, m)$. Let

$$f(n, x_n) = \exp\bigl[r(n) - \sum_{j=1}^{m} a_j(n) x_n(-k_j)\bigr].$$

Then $(C1)$ — $(C3)$ are satisfied. By Theorem 3, we get the following:

Corollary 2. *System (18) is permanent for all $k_j \in Z^+$ $(j = 1, \cdots, m)$.*

Remark 1. Corollary 2 generalizes Corollary 1. The result, biologically, means that system (18), in which the birth rate r and the coefficients a_j of the density dependence terms depending on time n, becomes permanent if only the birth rate and the density dependence are eventually positive and eventually bounded above, even if they are negative when time n is not so large.

(III) We will consider some other equations that are different from the Ricker type population models. Bellows [1] used the following equation to model the population of an insect:

$$x(n+1) = \frac{\lambda x(n)}{1 + [\beta x(n)]^\gamma} \qquad n = 0, 1, \cdots,$$

where λ, β, and γ are constants with $\lambda > 1$, $\beta > 0$, and $\gamma > 0$. Now, take some delays into consideration on the system as follows:

$$x(n+1) = \frac{\lambda x(n)}{1 + [\sum_{i=1}^m \beta_i x(n-k_i)]^\gamma} \qquad n = 0, 1, \cdots, \tag{19}$$

$$x(n+1) = \frac{\lambda x(n)}{1 + \sum_{i=1}^m [\beta_i x(n-k_i)]^{\gamma_i}} \qquad n = 0, 1, \cdots, \tag{20}$$

where β_i and γ_i are constants with $\beta_i \geq 0$, $\sum_{i=1}^m \beta_i > 0$, $\gamma_i > 0$, and $k_i \in Z^+$ $(i = 1, \cdots, m)$. Applying Theorem 3 to systems (19) and (20), we have the following:

Corollary 3. *Systems (19) and (20) are permanent for all $k_i \in Z^+$ $(i = 1, \cdots, m)$.*

Furthermore, for the two nonautonomous equations

$$x(n+1) = \frac{\lambda(n) x(n)}{1 + [\sum_{i=1}^m \beta_i(n) x(n-k_i)]^{\gamma(n)}}, \tag{21}$$

$$x(n+1) = \frac{\lambda(n) x(n)}{1 + \sum_{i=1}^m [\beta_i(n) x(n-k_i)]^{\gamma_i(n)}}, \tag{22}$$

where $\lambda(n)$ and $\beta_i(n)$ $(i = 1, \cdots, m)$ satisfy the eventual conditions

$$\begin{aligned}
1 < \lambda_1 \leq \liminf \lambda(n) \leq \limsup \lambda(n) \leq \lambda_2, \\
0 \leq \beta_{i1} \leq \liminf \beta_i(n) \leq \limsup \beta_i(n) \leq \beta_{i2}, \\
0 < \gamma_1 \leq \liminf \gamma(n) \leq \limsup \gamma(n) \leq \gamma_2, \\
0 < \gamma_{i1} \leq \liminf \gamma_i(n) \leq \limsup \gamma_i(n) \leq \gamma_{i2},
\end{aligned}$$

and $\sum_{i=1}^m \beta_{i1} > 0$, we get the following result:

Corollary 4. *Systems (21) and (22) are permanent for all $k_i \in Z^+$ $(i = 1, \cdots, m)$.*

4. Generalization of Theorem 2

In this section we will try to generalize Theorem 2. We consider the nonautonomous delay difference Kolmogorov-type population model

$$\begin{cases} x(n+1) = x(n)f(n, x_n, y_n) \\ y(n+1) = y(n)g(n, x_n, y_n), \quad n = 0, 1, \cdots \end{cases} \tag{23}$$

where we define for $s = -\nu, -\nu+1, \cdots, -1, 0$ ($\nu \in Z^+$),

$$x_n(s) = x(n+s) \qquad \text{and} \qquad y_n(s) = y(n+s).$$

The initial condition of (23) is given as

$$\begin{aligned} &x(s) \geq 0, \ \ y(s) \geq 0, \quad s = -\nu, -\nu+1, \cdots, 0, \\ &x(0) > 0, \ \ y(0) > 0. \end{aligned}$$

Define $X = \{\phi : \{-\nu, -\nu+1, \cdots, 0\} \to R^2_+\}$ and $f, g : Z^+ \times X \to R_+$. For each $\phi \in X$, the norm of ϕ is defined as $||\phi|| = \max\{|\phi(s)| \,\big|\, s = -\nu, -\nu+1, \cdots, -1, 0\}$, where $|\cdot|$ is any norm in R^2.

The assumptions of the functionals f and g are the following from $(H1)$ to $(H6)$:

$(H1)$ There exist constants $\delta_{1f} > 1$, $1 > \delta_{2f} > 0$, $K_{1f} > 0$, and $K_{2f} > 0$ ($K_{2f} \geq K_{1f}$) such that for all sufficiently large n,

$$\begin{aligned} &x(n+s) \in [0, K_{1f}],\ s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n, 0) > \delta_{1f}, \\ &x(n+s) \in [K_{2f}, \infty), s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n, 0) < \delta_{2f}. \end{aligned} \tag{24}$$

$(H2)$ For all sufficiently large n,

$$x_n, y_n \in X \Longrightarrow f(n, x_n, 0) \geq f(n, x_n, y_n). \tag{25}$$

Furthermore, for each $\bar{x} > 0$ and $\bar{y} > 0$, there exists an $1 > l_f(\bar{x}, \bar{y}) > 0$ such that for all sufficiently large n,

$$||x_n|| \leq \bar{x}, \ ||y_n|| \leq \bar{y} \Longrightarrow f(n, x_n, y_n) \geq l_f(\bar{x}, \bar{y}). \tag{26}$$

$(H3)$ There exists an $M_f > 1$ such that for all sufficiently large n,

$$x_n, y_n \in X \Longrightarrow f(n, x_n, y_n) \leq M_f. \tag{27}$$

$(H4)$ There exist constants $\delta_{1g} > 1$, $1 > \delta_{2g} > 0$, $K_{1g} > 0$, and $K_{2g} > 0$ ($K_{2g} \geq K_{1g}$) such that for all sufficiently large n,

$$\begin{aligned} &y(n+s) \in [0, K_{1g}],\ s = -\nu, \cdots, 0 \Longrightarrow g(n, 0, y_n) > \delta_{1g}, \\ &y(n+s) \in [K_{2g}, \infty), s = -\nu, \cdots, 0 \Longrightarrow g(n, 0, y_n) < \delta_{2g}. \end{aligned} \tag{28}$$

$(H5)$ For all sufficiently large n,

$$x_n, y_n \in X \Longrightarrow g(n, 0, y_n) \geq g(n, x_n, y_n). \tag{29}$$

Furthermore, for each $\bar{x} > 0$ and $\bar{y} > 0$, there exists an $1 > l_g(\bar{x}, \bar{y}) > 0$ such that for all sufficiently large n,

$$||x_n|| \leq \bar{x},\ ||y_n|| \leq \bar{y} \Longrightarrow g(n, x_n, y_n) \geq l_g(\bar{x}, \bar{y}). \tag{30}$$

$(H6)$ There exists an $M_g > 1$ such that for all sufficiently large n,

$$x_n, y_n \in X \Longrightarrow g(n, x_n, y_n) \leq M_g. \tag{31}$$

In the assumptions of f, propositions (24), (26), and (27) are similar to and correspond with (6), (7), and (8), respectively. $(H1)$ assumes that the growth rate for small population in the absence of competitors is positive, while there is a self-crowding effect creating a negative growth rate at high population levels, even in the absence of competitors. In $(H2)$, (25) states that the existence of y is negative to the growth of x. The relation (26) assumes that the negative fluctuation effect on the growth rate of x is limited for limited population densities of of species x and y. $(H3)$ assumes that there is an upper bound for the growth rate of x. Assumptions $(H4)$, $(H5)$, and $(H6)$ are the ones in which f in $(H1)$, $(H2)$, and $(H3)$ are replaced by g. When system (23) satisfies $(H1)$ through $(H6)$, we call it a *competition system.*

We obtain the following:

Theorem 4. *Suppose $(H1)$ — $(H6)$ hold. Let $\eta_x = K_{2f}M_f^{\nu+1}$, $\eta_y = K_{2g}M_g^{\nu+1}$, and $(x(n), y(n))$ be any solution of (23). Then for all large n,*

$$x(n) \leq \eta_x \qquad \text{and} \qquad y(n) \leq \eta_y.$$

Assume further that there is a $\delta_0 > 1$ such that for all sufficiently large n,

(i) $||x_n|| \leq \delta_0,\ ||y_n|| \leq \eta_y + \delta_0 \Longrightarrow f(n, x_n, y_n) > \delta_0,$

(ii) $||x_n|| \leq \eta_x + \delta_0,\ ||y_n|| \leq \delta_0 \Longrightarrow g(n, x_n, y_n) > \delta_0.$

Then system (23) is permanent.

(*Proof*) We will first prove the former of Theorem 4. We need only show that $x(n) \leq \eta_x$ for all large n since the case where $y(n) \leq \eta_y$ for all large n can be shown similarly.

It follows from $(H1)$ — $(H3)$ that there exists a sufficiently large number $N > 0$ such that the following $(H1)'$ — $(H3)'$ hold :

$(H1)'$ There exist constants $\delta_{1f} > 1$, $1 > \delta_{2f} > 0$, $K_{1f} > 0$, and $K_{2f} > 0$ $(K_{2f} \geq K_{1f})$ such that for all $n \geq N$,

$$\begin{aligned} &x(n+s) \in [0, K_{1f}],\ s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n, 0) > \delta_{1f}, \\ &x(n+s) \in [K_{2f}, \infty), s = -\nu, \cdots, 0 \Longrightarrow f(n, x_n, 0) < \delta_{2f}. \end{aligned} \tag{24'}$$

$(H2)'$ (*the former*) For all $n \geq N$,

$$x_n, y_n \in X \Longrightarrow f(n, x_n, 0) \geq f(n, x_n, y_n). \tag{25'}$$

$(H3)'$ There exists an $M_f > 1$ such that for all $n \geq N$,

$$x_n, y_n \in X \Longrightarrow f(n, x_n, y_n) \leq M_f. \tag{27'}$$

From (25') we have for all $n \geq N$,

$$x(n+1) \leq x(n)f(n, x_n, 0). \tag{32}$$

Thus, we can see by the same arguments as in the proof of Theorem 3 that there is a large $N_0 > N$ such that

$$x(N_0) \leq K_{2f} \quad \text{and} \quad x(N_0+1) \leq \eta_x. \tag{33}$$

We will show that $x(n) \leq \eta_x$ for all large n. Otherwise, it follows from (33) that there exist N_1 and N_2 ; $N_2 > N_1 > N$ such that

$$\begin{aligned} &x(N_1) \leq K_{2f}, \quad X(N_2) \leq \eta_x, \quad x(N_2+1) > \eta_x \\ &x(n) \in [K_{2f}, \eta_x] \quad \text{for } n = N_1+1, \cdots, N_2. \end{aligned} \tag{34}$$

By (27') and (33) we have for $n \geq N_1$,

$$x(n+1) \leq x(n)M_f \leq x(n-1)M_f^2 \leq \cdots \leq x(N_1)M_f^{n+1-N_1}.$$

When $n = N_2$,

$$\eta_x < x(N_2+1) \leq x(N_1)M_f^{N_2+1-N_1} \leq K_{2f}M_f^{N_2+1-N_1},$$

which implies $N_2 - N_1 > \nu$ since $\eta_x = K_{2f}M_f^{\nu+1}$ and $M_f > 1$. Therefore, $x(N_2+s) \in [K_{2f}, \infty)$ for $s = -\nu, \cdots, 0$. By (24') and (32) we have

$$x(N_2+1) \leq x(N_2)f(N_2, x_{N_2}, 0) < x(N_2),$$

which is a contradiction to (34).

Next, we will show the latter of Theorem 4. An approach similar to the proof of Theorem 1 is adopted. Let

$$\bar{\eta_x} = \delta_0 l_f(\eta_x, \eta_y)^{\nu+1} \quad \text{and} \quad \bar{\eta_y} = \delta_0 l_g(\eta_x, \eta_y)^{\nu+1}.$$

We prove here that

$$x(n) \geq \bar{\eta_x} \quad \text{for all large } n. \tag{35}$$

The proof that $y(n) \geq \bar{\eta_y}$ for all large n is similar. Since $x(n) \leq \eta_x$ and $y(n) \leq \eta_y$ for all large n, we have from (26) that there exists an $N' > N$ such that for all $n \geq N'$,

$$f(n, x_n, y_n) \geq l_f(\eta_x, \eta_y). \tag{36}$$

From (i) we see that there is an $N_0' > N'$ such that

$$x(N_0') \geq \delta_0 \quad \text{and} \quad x(N_0'+1) \geq \bar{\eta_x}. \tag{37}$$

We need only prove $x(N_0') \geq \delta_0$ since it is easy to get $x(N_0'+1) \geq \bar{\eta_x}$ if $x(N_0') \geq \delta_0$ by (36). If (37) is false, then we have $x(n) < \delta_0$ for all large n. From (i) we obtain for all large n,

$$x(n+1) > x(n)\delta_0.$$

This implies $x(n) \to +\infty$ as $n \to +\infty$, which contradicts that $x(n) < \delta_0$ for all large n.

If (35) is false, then from (37) there exist N_1' and N_2' ; $N_2' > N_1' > N'$ such that

$$\begin{aligned} &x(N_1') \geq \delta_0, \quad x(N_2') \geq \bar{\eta_x}, \quad x(N_2'+1) < \bar{\eta_x} \\ &x(n) \in [\bar{\eta_x}, \delta_0] \quad \text{for } n = N_1'+1, \cdots, N_2'. \end{aligned} \tag{38}$$

From (36) we have

$$\begin{aligned} \bar{\eta_x} > x(N_2'+1) &\geq x(N_2')l_f(\eta_x,\eta_y) \geq x(N_2'-1)l_f(\eta_x,\eta_y)^2 \\ &\geq \cdots \geq x(N_1')l_f(\eta_x,\eta_y)^{N_2'+1-N_1'}, \end{aligned}$$

which implies $N_2' - N_1' > \nu$ since $\bar{\eta_x} = \delta_0 l_f(\eta_x,\eta_y)^{\nu+1}$ and $0 < l_f(\eta_x,\eta_y) < 1$. Thus, $x(N_2'+s) \in [0,\delta_0]$ for $s = -\nu, -\nu+1, \cdots, 0$. By (i), we have

$$x(N_2'+1) = x(N_2')f(N_2', x_{N_2'}, y_{N_2'}) > x(N_2')\delta_0 > x(N_2'),$$

which is a contradiction to (38). The proof of Theorem 4 is thus completed.

5. A corollary of Theorem 4 and future work

We apply Theorem 4 to system (4). Let

$$\begin{aligned} f(n,x_n,y_n) &= \exp[r_1 - a_{11}x_n(-k_1) - a_{12}y_n(-k_2)], \\ g(n,x_n,y_n) &= \exp[r_2 - a_{21}x_n(-l_1) - a_{22}y_n(-l_2)]. \end{aligned}$$

Then $(H1)$ — $(H6)$ are satisfied. By Theorem 4, the following holds:

Corollary 5. *System (4) is permanent if*

$$r_2a_{11} - r_1a_{21}e^{r_2(l_2+1)} > 0 \quad \textit{and} \quad r_1a_{22} - r_2a_{12}e^{r_1(k_1+1)} > 0$$

hold.

We can see that there is a gap between the conditons of Corollary 5 and Theorem 2. Theorem 4 has not generalized Theorem 2 and has room for improvement, because it is considered that conditions (i) and (ii) in Theorem 4 are too strong. They should be made weaker for the permanence of (23), which is left for future work.

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